Journal of Engineering Mathematics, Vol. 11, No. 4, October 1977
Noordhoff International Publishing–Leyden
Printed in The Netherlands

A comparative study of elasticity, shell and boundary layer solutions applied to axially compressed cylinders

S. MIRZA

Department of Mechanical Engineering University of Ottawa, Ottawa, Canada

J. C. RAJPUT

Pratt and Whitney Aircraft, Montreal, Canada

(Received June 22, 1976)

SUMMARY
This paper deals primarily with a comparative study based on different methods of solution for the problem of axially compressed cylinders. A comprehensive discussion on the range of validity of these types of solutions is also included, and an extensive numerical analysis has been carried out.

1. Introduction

The study of hollow circular cylinders has been of considerable interest and of much practical importance because of their numerous uses in the industry. A detailed study of solid cylinders under practical systems of loading was reported by Filon [2]. This method was improved by Pickett [3], who introduced the multiple Fourier method in solving these problems. In the multiple Fourier method, two or more series of particular solutions are chosen. Another type of solution for problems of finite cylinders in terms of displacement functions, has been given by Valov [4] and Blair and Veeder [5]. Some of these solutions have been extended to the case of hollow cylinders [6, 7]. The analysis in [6] provides only a formal mathematical solution and does not deal with the numerical results. The three-dimensional elasticity solution involves a great deal of labor and computational difficulties.

Johnson and Reissner [8] founded a theory of thin elastic cylindrical shells in which the solutions are obtained by asymptotic integration of the equations of three-dimensional theory of elasticity. This solution is given for a semi-infinite cylindrical shell. Subsequently, Reiss [9] showed that the solution obtained in [8] represents the exact solution in the regions away from the edge, that is, in the interior of the shell. Further, he pointed out that the solution in [8] is not valid in the narrow edge zones recognized as boundary layer. The problem of axi-symmetric compression of hollow cylinders was solved by Widera and Wu [10] by the superposition of shell and boundary layer solutions on the interior solution. The solution in [10] is given for orthotropic cylindrical shells. But due to the nature of the particular integrals chosen in [10], the solution presented in that paper is not valid for isotropic cylinders.

The purpose of this paper is two-fold: to rectify the solution in [10], in particular for isotropic cylinders, and to provide a comprehensive numerical solution for the axially compressed hollow cylinders based on various theories available in the literature. The functions and the series involved in the solutions present considerable difficulty in computations and it is partly due to this reason that no real comparative data is available. The latter goal, mentioned above, provides the basis for comparison, in the engineering sense, between the elasticity, shell and boundary layer solutions. This is done with a view to recommend guidelines for design engineers.

2. Boundary conditions

The boundary conditions for the elasticity, shell and boundary layer solutions are written separately as follows.

(i) Three-dimensional elasticity solution:

The cylinder having an inner and outer radii of a, b and a half length c is subjected to an axial load of P. The curved surfaces of the cylinder are traction free and the ends have a constant axial displacement w. The radial displacement u at the ends is taken as zero. Thus,

$$\sigma_r = \tau_{rz} = 0 \text{ at } r = a, b,$$
$$u = 0 \text{ and } w = \pm k \text{ at } z = \pm c. \tag{2.1}$$

The origin of the cylindrical coordinate system r, θ, z is located at mid-height. Here the constant k is chosen such that,

$$\int_a^b \sigma_z|_{z=c} \, 2\pi r dr = -P. \tag{2.2}$$

The normal and shear stresses are given by $\sigma_r, \sigma_z, \tau_{rz}$ etc.

(ii) Shell and boundary layer solution:

For this type of solution, it is found to be convenient to change the origin of the coordinates to the mid-surface of the shell. Hence, in this case, the boundary conditions are

$$\sigma_r = \tau_{rz} = 0 \text{ at } r = a_1 \pm h,$$
$$u = 0 \text{ and } w = \pm k \text{ at } z = \pm c. \tag{2.3}$$

In these equations a_1 and h represent the mid-surface radius and half thickness of shell. The constant k is again chosen from Eq. (2.2) with the upper and lower limits of integration changed to $a_1 + h$ and $a_1 - h$ respectively.

3. Elasticity and shell and boundary layer solutions

The two types of solutions are first discussed in the following. The mathematical form of the elasticity solution, which has been reported earlier [6], is being mentioned here for the sake of completeness of the discussion.

(i) Elasticity solution:

The ϕ-component of the Galerkin vector is taken as

$$\phi = -Lz^3 + \sum_{m=1}^{\infty} \frac{1}{\alpha_m^2} [\alpha_m z \cosh \alpha_m z$$

$$- (1 + \alpha_m c \tanh \alpha_m c) \sinh \alpha_m z][E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)]$$

$$+ \sum_{n=1}^{\infty} \frac{\sin \beta_n z}{\beta_n^3} [A_n i\beta_n r J_1(i\beta_n r) - B_n J_0(i\beta_n r)$$

$$+ C_n \beta_n r H_1^1(i\beta_n r) + D_n i H_0^1(i\beta_n r)], \tag{3.1}$$

where, $A_n, B_n, \ldots, \alpha_m, \beta_n$ and L are unknown constants. Also, J, Y, are Bessel functions of the first and second kind and H^1 are Hankel functions of the first kind. Using basic definitions of stresses and displacements in terms of ϕ, [1], the following equations can be established:

$$2Gu = \sum_{n=1}^{\infty} \frac{\cos \beta_n z}{\beta_n} [A_n \beta_n r J_0(i\beta_n r) + B_n i J_1(i\beta_n r)$$

$$- C_n i\beta_n r H_0^1(i\beta_n r) - D_n H_1^1(i\beta_n r)]$$

$$+ \sum_{m=1}^{\infty} \frac{1}{\alpha_m} [\alpha_m z \sinh \alpha_m z - \alpha_m c \tanh \alpha_m c \cosh \alpha_m z][E_m J_1(\alpha_m r) + F_m Y_1(\alpha_m r)]$$

$$2Gw = -6(1 - 2\nu)Lz + \sum_{n=1}^{\infty} \frac{\sin \beta_n z}{\beta_n} \{A_n[-4(1 - \nu)J_0(i\beta_n r)$$

$$+ i\beta_n r J_1(i\beta_n r)] + B_n J_0(i\beta_n r) + C_n[4(1 - \nu) i H_0^1(i\beta_n r)$$

$$+ \beta_n r H_1^1(i\beta_n r)] + D_n i H_0^1(i\beta_n r)\} - \sum_{m=1}^{\infty} \frac{1}{\alpha_m} [\alpha_m z \cosh \alpha_m z$$

$$- (3 - 4\nu + \alpha_m c \tanh \alpha_m c) \sinh \alpha_m z][E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)],$$

$$\sigma_z = -6(1 - \nu)L + \sum_{n=1}^{\infty} \cos \beta_n z \{A_n[-2(2 - \nu)J_0(i\beta_n r)$$

$$+ i\beta_n r J_1(i\beta_n r)] + B_n J_0(i\beta_n r) + C_n[2(2 - \nu) i H_0^1(i\beta_n r)$$

$$+ \beta_n r H_1^1(i\beta_n r)] + D_n i H_0^1(i\beta_n r)\} + \sum_{m=1}^{\infty} [-\alpha_m z \sinh \alpha_m z$$

$$+ (2 - 2\nu + \alpha_m c \tanh \alpha_m c) \cosh \alpha_m z][E_m J_0(\alpha_m r) + F_m Y_0(\alpha_m r)], \tag{3.2}$$

where G and ν designate the shear modulus and Poisson's ratio of the material. The constants $A_n, B_n, \ldots$ are evaluated from the satisfaction of the boundary conditions given by Eqs. (2.1) and (2.2). These equations are further discussed in section (6).

(ii) Shell and boundary layer solution:

The state of stress in this case is obtained by the superposition of an elementary solution and a shell type solution. The elementary solution is [10]:

$$\hat{\sigma}_z = -KE/c, \quad w = Kz/c, \quad u = Kvr/c, \tag{3.3}$$

where the constant $K = Pc/(4\pi a_1 hE)$.

The basic equations for the second solution are obtained by the asymptotic integration of the equilibrium equations, written in cylindrical coordinates, along with the generalized Hooke's law and strain displacement equations [10]. In this solution the boundary conditions are taken as

$$w = 0, \quad u = u \ \text{ at } \ z = 0.$$

The quantities s_r, s_z, etc., represent the non-dimensional stresses corresponding to σ_r, σ_z, etc. The non-dimensional displacements v_r and v_z are defined as

$$v_r = u/q, \quad v_z = w/q, \quad q = a_1\sigma(1 - v^2)/E. \tag{3.4}$$

The governing equations for interior solution are obtained by choosing $\mu = \lambda^{\frac{1}{2}}$.

The second choice, $\mu = \lambda$, yields the boundary layer equations. The parameters μ and λ are defined as $\mu = L/a$ and $\lambda = h/a$, where L is a small and undetermined length scale, denoting the boundary layer. For further explanation and discussion of these parameters we refer to [10].

For the interior and boundary layer solutions the variable z is non-dimensionalized as ξ and η respectively,

$$\xi = (z + c)/\sqrt{ha_1}, \quad \eta = (z + c)/h,$$

and the variable r is non-dimensionalized as $\rho = (r - a_1)/h$. Asymptotic expansions for stress and displacement components for the two cases are written as power series of $\lambda^{\frac{1}{2}}$. The functions $s^{(i)}(\xi, \rho)$ and $v^{(i)}(\xi, \rho)$ represent the stress and displacement for the interior solution and $t^{(i)}(\eta, \rho)$ and $w^{(i)}(\eta, \rho)$ give the boundary layer solution. Equating the coefficients of $\lambda^0, \lambda^{\frac{1}{2}}, \lambda^1$ etc., on both the sides of equations and integrating in a step by step manner, a solution for the interior problem can be obtained [8]:

$$\begin{aligned} &v_r = v_r^{(0)} + v_r^{(2)}\lambda, \quad v_z = v_z^{(1)}\lambda^{\frac{1}{2}} + v_z^{(3)}\lambda^{\frac{3}{2}}, \\ &s_\theta = s_\theta^{(0)} + s_\theta^{(2)}\lambda, \quad s_z = s_z^{(0)} + s_z^{(2)}\lambda, \quad s_r = s_r^{(2)}\lambda + s_r^{(4)}\lambda^2, \ \text{etc.} \end{aligned} \tag{3.5}$$

For the boundary layer solution near the lower edge, $z = -c$, a procedure similar to the one described above yields the first system of differential equations. Assuming the form of displacements as

$$w_z^{(2)}(\eta, \rho) = \sum_{n=1}^{\infty} a_n u_1^n(\rho, \beta_n^*) \exp(-\beta_n^* \eta),$$

$$w_r^{(2)}(\eta, \rho) = \sum_{n=1}^{\infty} a_n u_2^n(\rho, \beta_n^*) \exp(-\beta_n^* \eta), \tag{3.6}$$

and inserting them back in the first system equations finally yields

$$t_z^{(0)}(\eta, \rho) = \sum_{n=1}^{\infty} a_n \tau_{11}^n(\rho, \beta_n^*) \exp(-\beta_n^* \eta),$$

$$t_r^{(0)}(\eta, \rho) = \sum_{n=1}^{\infty} a_n \tau_{22}^n(\rho, \beta_n^*) \exp(-\beta_n^* \eta),$$

$$t_{rz}^{(0)}(\eta, \rho) = \sum_{n=1}^{\infty} a_n \tau_{12}^n(\rho, \beta_n^*) \exp(-\beta_n^* \eta),$$

$$t_\theta^{(0)}(\eta, \rho) = \sum_{n=1}^{\infty} a_n \tau_{33}^n(\rho, \beta_n^*) \exp(-\beta_n^* \eta),$$

$$\ddot{u}_1^n \pm A^* \beta_n^{*2} u_1^n - B^* \beta_n^* \dot{u}_2^n = 0,$$
$$\ddot{u}_2^n + \beta_n^{*2} u_2^n / 2R - B^{**} \beta_n^* \dot{u}_1^n = 0. \qquad (3.7)$$

The quantities τ_{ij}, A^*, B^*, etc. are

$$\tau_{11}^n = -\beta_n^* u_1^n \{1 + \nu^2 R/(1-\nu)\} - \nu R \dot{u}_2^n, \quad \tau_{22}^n = \{(1-\nu)\dot{u}_2^n - \beta_n^* \nu u_1^n\} R,$$
$$\tau_{12}^n = (\dot{u}_1^n - \beta_n^* u_2^n)(1-\nu)/2, \quad \tau_{33}^n = -\beta_n^* u_1^n \{\nu + \nu^2 R/(1-\nu)\} + \dot{u}_2^n \nu,$$
$$A^* = 2(1 - \nu + \nu^2 R)/(1-\nu)^2, \quad B^* = (2R\nu + 1 - \nu)/(1-\nu),$$
$$B^{**} = (2R\nu + 1 - \nu)/\{2R(1-\nu)\} \text{ and } R = (1-\nu^2)/(1 + \nu - 2\nu^2).$$

The notation $\dot{u}_1$, $\ddot{u}_1$ means the first or the second derivative of u_1 with respect to ρ. In these equations, a_n and β_n^* are complex constants obtained from the boundary conditions. Boundary conditions (*ii*) are to be satisfied at $\eta = \xi = 0$ by the interior and boundary layer solutions together. Without any loss of generality, the mid-surface displacements $V_r^{(2)}(\xi, 0)$ and $V_z^{(3)}(\xi, 0)$ which are arbitrary functions of ξ can be taken as zero at $\xi = 0$.

The solutions for the variables u_1 and u_2, which are introduced in Eqs. (3.7) and which govern the complete boundary layer solution, are not correctly presented in reference [10]. The modified solutions are presented as follows. Equations (3.7) e and f can be combined to give a single fourth order differential equation which is found to have repeated roots. Hence the solution is taken in the form

$$u_1^n = (A_2 + \rho B_2) \cos \beta_n^* \rho + (C_2 + \rho D_2) \sin \beta_n^* \rho. \qquad (3.8)$$

Substitution of u_1^n in Eq. (3.7) yields

$$u_2^n = \{(3-4\nu)B_2/\beta_n^* - C_2 - \rho D_2 \cos \beta_n^* \rho + \{(3-4\nu)D_2/\beta_n^* + A_2 - \rho C_2\} \sin \beta_n^* \rho. \qquad (3.8)$$

For convenience u_1^n and u_2^n are separated into symmetric and anti-symmetric parts as follows:

$$u_1^n = A_2 \cos \beta_n^* \rho + \rho D_2 \sin \beta_n^* \rho, \; u_2^n = \{(3-4\nu)D_2/\beta_n^* + A_2\} \sin \beta_n^* \rho - \rho D_2 \cos \beta_n^* \rho,$$

and

$$u_1^n = C_2 \sin \beta_n^* \rho + \rho B_2 \cos \beta_n^* \rho, \; u_2^n = \{(3-4\nu)B_2/\beta_n^* - C_2\} \cos \beta_n^* \rho - \rho B_2 \sin \beta_n^* \rho. \qquad (3.9)$$

Substituting Eqs. (3.9) along with the no-stress conditions on the cylindrical surface $t_r^0(\eta, \pm 1) = 0$ and $t_{rz}^0(\eta, \pm 1) = 0$, into the basic differential equations, the following equations are obtained:

$$\sin 2\beta_n^* + 2\beta_n^* = 0, \quad \sin 2\beta_n^* - 2\beta_n^* = 0. \tag{a}$$

The final expressions for the symmetric and anti-symmetric solutions u_1^n and u_2^n are

$$u_1^n = K_n \cos \beta_{n_1}\rho + \rho \sin \beta_{n_1}\rho, \quad u_2^n = \{3 - 4\nu)/\beta_{n_1} + K_n\} \sin \beta_{n_1}\rho - \rho \cos \beta_{n_1}\rho,$$
$$u_1^n = p_n \sin \beta_{n_2}\rho + \rho \cos \beta_{n_2}\rho, \quad u_2^n = \{(3 - 4\nu)/\beta_{n_2} - p_n\} \cos \beta_{n_2}\rho + \rho \sin \beta_{n_2}\rho, \tag{3.10}$$

where,

$$K_n = -2(1 - \nu)/\beta_{n_1} - \tan \beta_{n_1}, \quad p_n = 2(1 - \nu)/\beta_{n_2} - \cot \beta_{n_2}.$$

In the above equations, β_{n_1} and β_{n_2} are the roots of Eqs. (a). The complex constants a_n are now evaluated by using the principle of variation of energy due to complex stress field. It can be seen that a_n must satisfy the following system of equations.

$$\sum_{m=1}^{\infty} (X_{mn} + Y_{mn})a_n = b_m. \tag{3.11}$$

The quantities X_{mn}, Y_{mn} and b_m are given as

$$\begin{aligned}
X_{mn}/(1 - \nu) &= \sin \delta_1\{(2\nu - \bar{K}_m\bar{\beta}_{m1})(K_n + 1/\delta_1) - \bar{\beta}_{m1}(K_n/\delta_1 - 1 + 2/\delta_1^2)\}/\delta_1 \\
&+ \sin \delta_2\{(2\nu - \bar{K}_m\bar{\beta}_{m1})(K_n - 1/\delta_2) - \beta_{m1}(K_n/\delta_2 + 1 - 2/\delta_2^2)\}/\delta_2 \\
&+ \sin \delta_3\{(2\nu + \bar{p}_m\bar{\beta}_{m2})(p_n - 1/\delta_3) - \bar{\beta}_{m2}(p_n/\delta_3 + 1 - 2/\delta_3^2)\}/\delta_3 \\
&+ \sin \delta_4\{-(2\nu + \bar{p}_m\bar{\beta}_{m2})(p_n + 1/\delta_4) + \bar{\beta}_{m2}(p_n/\delta_4 - 1 + 2/\delta_4^2)\}/\delta_4 \\
&+ \cos \delta_1\{-(2\nu - \bar{K}_m\bar{\beta}_{m1}) + \bar{\beta}_{m1}(K_n + 2/\delta_1)\}/\delta_1 + \cos \delta_2\{(2\nu - \bar{K}_m\bar{\beta}_{m1}) \\
&+ \bar{\beta}_{m1}(K_n - 2/\delta_2)\}/\delta_2 + \cos \delta_3\{(2\nu + \bar{p}_m\bar{\beta}_{m2}) + \bar{\beta}_{m2}(p_n - 2/\delta_3)\}/\delta_3 \\
&+ \cos \delta_4\{(2\nu + \bar{p}_m\bar{\beta}_{m2}) - \bar{\beta}_{m2}(p_n + 2/\delta_4)\}/\delta_4,
\end{aligned}$$

$$\begin{aligned}
b_m = &-4\nu/(\bar{\beta}_{m1} \cos \bar{\beta}_{m1}) - \nu^2\gamma^2\{\sin \bar{\beta}_{m2}[\bar{p}_m\bar{\beta}_{m2} - (1 - 2\nu)](2/\bar{\beta}_{m2} - 4/\bar{\beta}_{m2}^3) \\
&- \bar{\beta}_{m2}[6/\beta_{m2}^2 - 12/\beta_{m2}^4] + \cos \bar{\beta}_{m2}[4/\beta_{m2}^2(\bar{p}_m\bar{\beta}_{m2}(\bar{p}_m\bar{\beta}_{m2} - 1 + 2\nu) \\
&- \bar{\beta}_{m2}(12/\beta_{m2}^3 - 2/\beta_{m2})]\}/(1 - \nu^2),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
Y_{mn}/(1 - \nu) &= \sin \delta_1\{(\bar{K}_m\bar{\beta}_{m1} + 1 - 2\nu)[(3 - 4\nu)/\beta_{n1} + K_n + 1/\delta_1] \\
&+ \bar{\beta}_{m1}[((3 - 4\nu)/\beta_{n1} + K_n)/\delta_1 - 1 + 2/\delta_1^2]\}/\delta_1 \\
&+ \sin \delta_2\{-(\bar{K}_m\bar{\beta}_{m1} + 1 - 2\nu)[(3 - 4\nu)/\beta_{n1} + K_n - 1/\delta_2] \\
&+ \bar{\beta}_{m1}[-((3 - 4\nu)/\beta_{n1} + K_n)/\delta_2 - 1 + 2/\delta_2^2]\}/\delta_2 \\
&+ \sin \delta_3\{(\bar{p}_m\bar{\beta}_{m2} - 1 + 2\nu)[(3 - 4\nu)/\beta_{n2} - p_n + 1/\delta_3] \\
&- \bar{\beta}_{m2}[((3 - 4\nu)/\beta_{n2} - p_n)/\delta_3 - 1 + 2/\delta_3^2]/\delta_3 \\
&+ \sin \delta_4\{(\bar{p}_m\bar{\beta}_{m2} - 1 + 2\nu)[(3 - 4\nu)/\beta_{n2} - p_n - 1/\delta_4]
\end{aligned}$$

$$- \bar{\beta}_{m2}[((3 - 4\nu)/\beta_{n2} - p_n)/\delta_4 + 1 - 2/\delta_4^2]\}/\delta_4$$
$$+ \cos \delta_1\{-(\bar{K}_m\bar{\beta}_{m1} + 1 - 2\nu) - \bar{\beta}_{m1}[(3 - 4\nu)/\beta_{n1} + K_n + 2/\delta_1]\}/\delta_1$$
$$+ \cos \delta_2\{-(\bar{K}_m\bar{\beta}_{m1} + 1 - 2\nu) + \bar{\beta}_{m1}[(3 - 4\nu)/\beta_{n1} + K_n - 2/\delta_2]\}/\delta_2$$
$$+ \cos \delta_3\{-(\bar{p}_m\bar{\beta}_{m2} - 1 + 2\nu) + \bar{\beta}_{m2}[(3 - 4\nu)/\beta_{n2} - p_n + 2/\delta_3]\}/\delta_3$$
$$+ \cos \delta_4\{(\bar{p}_m\bar{\beta}_{m2} - 1 + 2\nu) - \bar{\beta}_{m2}[-(3 - 4\nu)/\beta_{n2} + p_n + 2/\delta_4]\}/\delta_4,$$

and

$$\delta_{1,2} = \bar{\beta}_{m1} \pm \beta_{n1}; \quad \delta_{3,4} = \bar{\beta}_{m2} \pm \beta_{n2}; \quad \gamma^4 = 3(1 - \nu^2)/4.$$

The notation $\overline{(\ )}$ represents the complex conjugate of ( ).

4. Computation and discussion of results

Numerical computations for the two theoretical cases were carried out on an IBM computer system with double precision. For the computational purpose the following inner to outer radius and half length to outer radius ratios were studied.

$$\frac{a}{b} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}; \quad \frac{c}{b} = 1, 2, 3$$

An experimental investigation was also conducted by the authors [16], and some of relevant data is shown in graphs. The details of the experiments performed are omitted from this paper.

Elasticity solution: It has been observed that there is a small residual stress σ_r at the curved surfaces $r = a, b$. This is partly due to the contradiction in the boundary requirements at the intersection of curved surfaces and the planes $z = \pm c$. According to the boundary conditions the radial displacement is zero at the ends. In the present case this condition leads to radial and tangential strains being zero. Writing these strains in terms of three-dimensional stress state and solving for σ_r and σ_θ results in the two relations

$$\sigma_r = \nu\sigma_z/(1 - \nu), \quad \sigma_\theta = \nu\sigma_z/(1 - \nu). \tag{4.1}$$

It is evident from Eqs. (4.1) that σ_r will not be equal to zero at $z = \pm c$, since σ_z is finite.

In the numerical analysis and solution of the infinite equations resulting from the application of boundary conditions and Fourier transforms, some very interesting observations were made regarding its convergence. For the thickest and shortest cylinder considered in this analysis, as many as 15 terms were used in the series. It was observed that the solutions degenerate when a large number of terms in the series are taken. These effects were predominant at the ends. On the contrary, with a large number of terms used in the series, a very slow convergence was observed near the middle of the cylinder. An explanation of this behavior can be obtained from the analysis of the infinite system. The system of equation can be written as

$$x_i = \sum_{k=1}^{\infty} C_{i,k}x_k + b_i \qquad (i = 1, 2, \ldots) \tag{4.2}$$

From the theory of infinite systems of linear simultaneous equations, the approximate solution obtained by using the method of reduction converges to the solution of the infinite system provided the system is regular and the free terms of it are bounded. An infinite system of the form (4.2) is regular if the sum of the absolute values of the coefficients of each row is less than unity:

$$\sum |c_{i,k}| < 1, \quad i = 1, 2, \ldots \tag{4.3}$$

For a convergent system, the free terms are bounded in the form,

$$|b_i| \leqq Q_1[1 - \sum |c_{i,k}|], \quad i = 1, 2, \ldots \tag{4.4}$$

where the constant $Q_1 > 0$. A numerical procedure was adopted to check the regularity of the system. Fifteen terms were considered and the resulting equations assembled in the form of Eq. (4.2). The sum of the moduli of the off-diagonal elements in each row of the coefficient matrix was obtained, and it was observed that Eqs. (4.3) are not satisfied. It is, therefore, concluded that the system is not regular and the convergence is not guaranteed.

Boundary layer solution: The infinite system (3.11) was solved by reducing it to a 10 by 10 system.

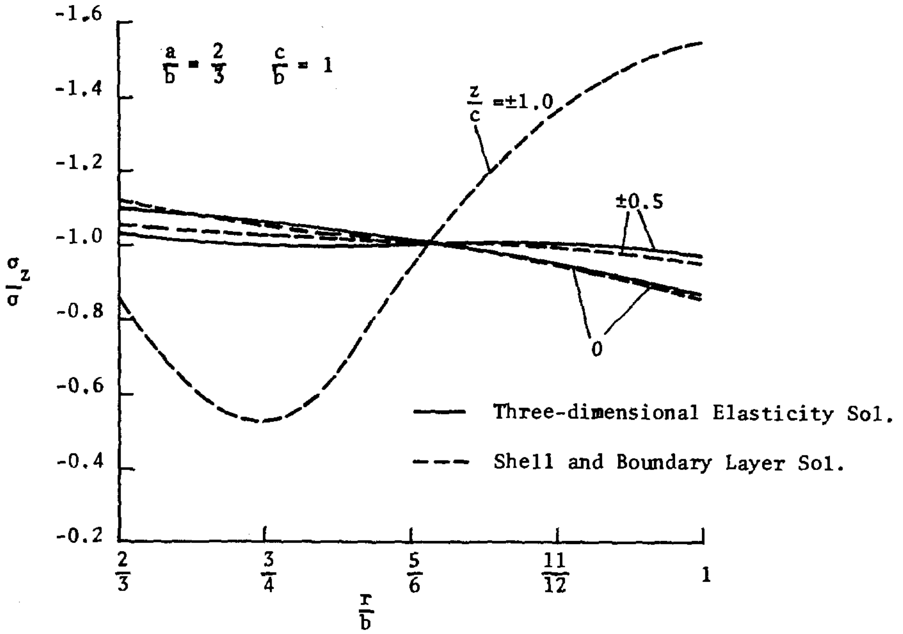
The matrix $(X_{mn} + Y_{mn})$ is a hermitian matrix and in terms of the criteria (4.3) and (4.4) the system is regular. The use of the first ten terms shows an indication of definite convergence. The roots β_{n1} and β_{n2}, as given in [12, 13], have been used in the calculations. In the boundary layer solution only the first non-zero system has been considered.

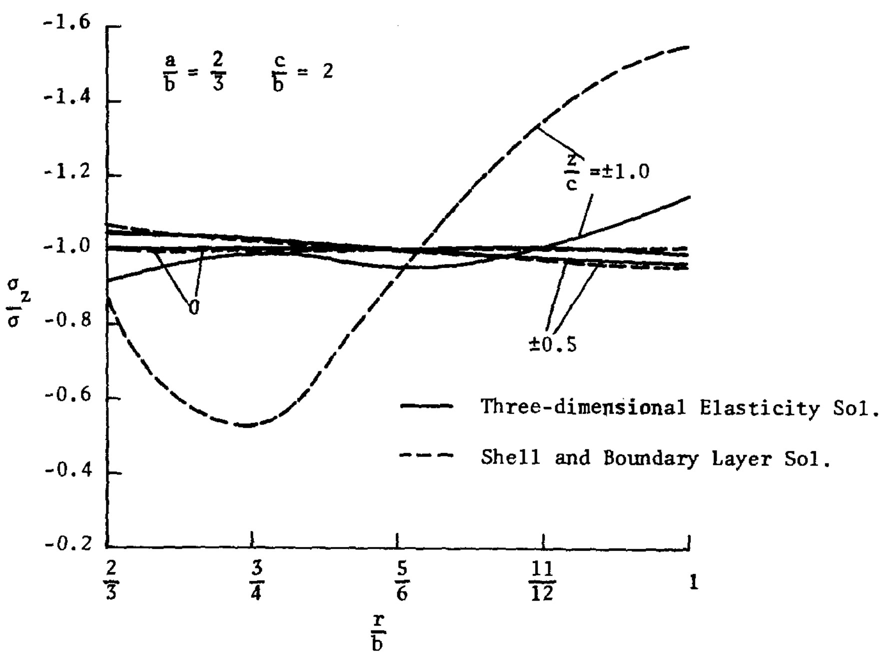
All the graphs generated from the numerical computations are not being presented in this report. The conclusions are, however, based on all the results. Figures 1, 2, 3 and 4 show the distribution of longitudinal stress obtained from the elasticity and shell and boundary layer solutions at various z/c ratios. A comparison of the longitudinal stress is shown in Figure 5 and 6. This variation is plotted against z/c. Figures 7 and 8 show the comparison of the radial displacement.

The two solutions give closer results in the regions away from the ends. For longer and thinner cylinders ($c/b \geqq 2$ and $a/b \leqq \frac{2}{3}$) both the solutions differ very slightly from each other. The maximum variation in the prediction from these two theories is about 27% of the average stress σ for a cylinder with $a/b = \frac{2}{3}$ and $c/b = 1$. This disparity occurs primarily in the edge zones. The minimum is about 2.5% for a cylinder with $a/b = \frac{7}{8}$ and $c/b = 1$. The variations of the radial and axial displacement, not shown here, also follow the same general pattern.

Comparison is made on the basis of how close the experimental variation is to any of the two solutions. On this basis it can be concluded that the three-dimensional theory gives better results for thick and short cylinders ($a/b \leqq \frac{1}{2}$ and $c/b \leqq 3$). While the shell and boundary layer solution should be preferred for thin and long cylinders ($a/b \geqq \frac{3}{4}$ and $c/b \geqq 2$). In the intermediate range of the parameters a/b, c/b and in the regions away from the end of the cylinder both the theories give similar results.

It can be seen that by properly accounting for the edge effects, shell theory gives good results for relatively thick shells ($a/b = 0.75$). The results obtained from classical shell theory, as reported earlier [15], were restricted to $a/b = 0.9$.





Figures 1, 2. Distribution of longitudinal stress.

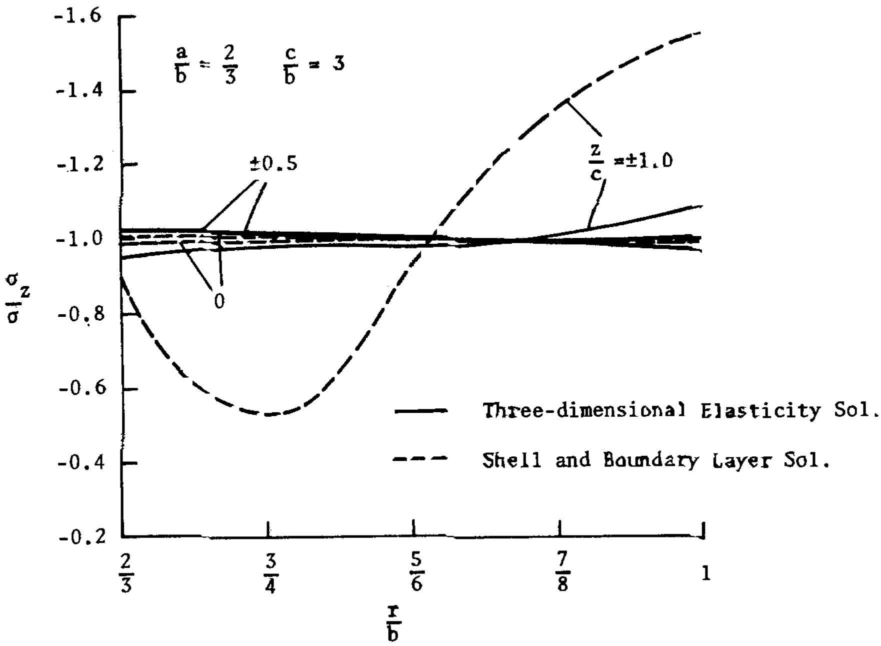


Figure 3. Distribution of longitudinal stress.

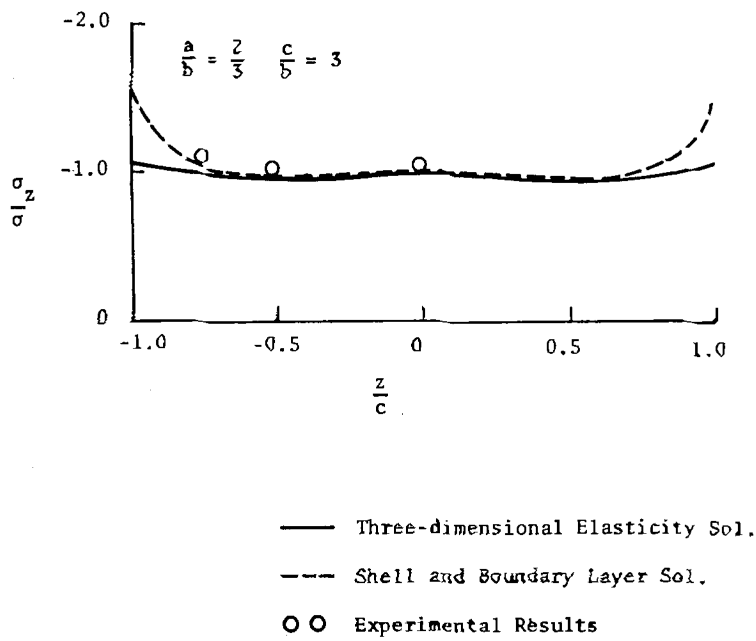


Figure 4. Distribution of longitudinal stress along other surface.

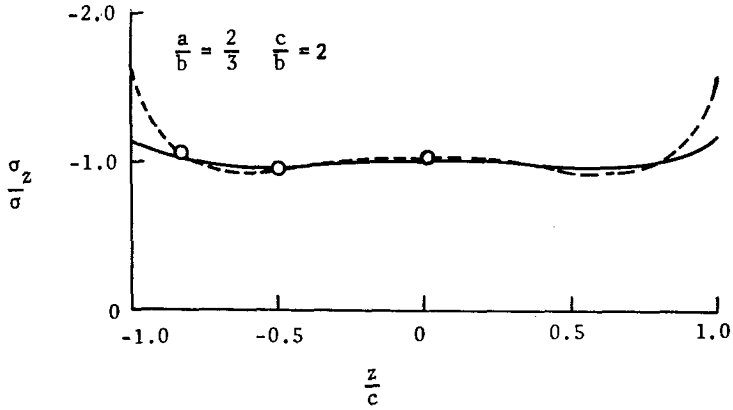
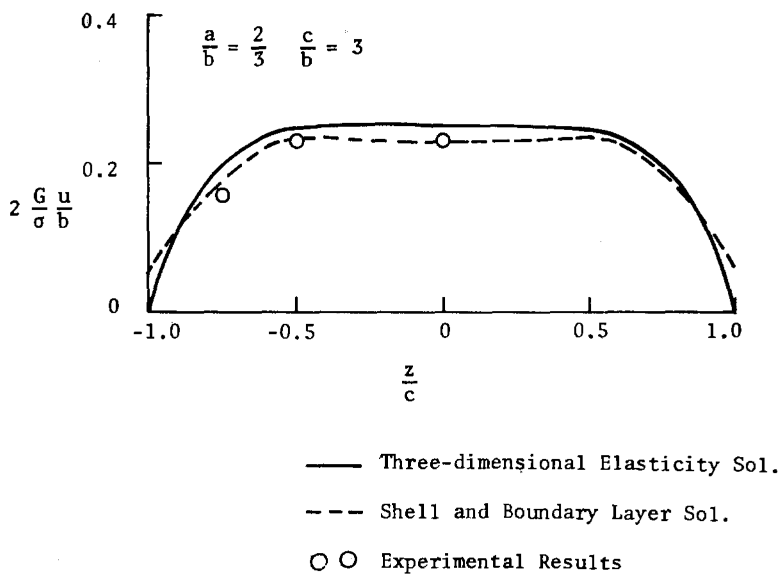
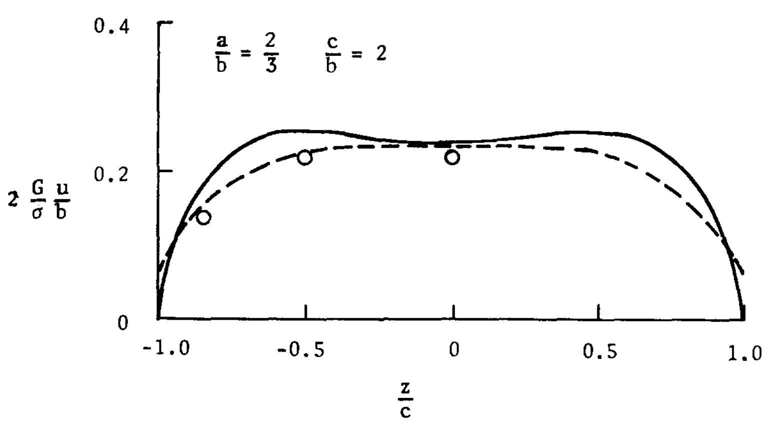


Figure 5. Distribution of longitudinal stress along the outer surface.





Figures 6, 7. Variation of radial displacement along the outer surface.

Acknowledgement

The NRC financial grant received for this project is gratefully acknowledged.

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